**Q1:** Define a sequence by and for

1. Show that for all

Since the sequence is defined recursively, the pair of inequalities can be proven inductively. Firstly, the base case holds for by Secondly, assume that holds. Then

and

as (per the inductive hypothesis). Thus, proving the statement for all by induction.

1. Show that is decreasing.

By looking at the difference between consecutive terms, we have

as . Therefore, is decreasing.

1. Prove that converges and find its limit.

Since is decreasing and bounded below, it converges by the monotone convergence theorem to some number As for all *n*, we have by the order properties of limits. Additionally, as as taking limits in the recursion formula gives

Thus, or Since by the previous statement, as required.

**Q2:** Determine whether the series

converges or diverges.

Let Since and Now let By the limit ratio test,

as by the definition of Thus, converges. By the comparison test, since

is true and converges, also converges.

**Q3:**

1. Explain why for each

Let us look at the RHS of the inequality with a geometric point of view. Looking at graph A, the function is strictly decreasing and continuous in the first quadrant, and the integral is a computation of the area below the graph between the values and Since the function is decreasing, the area can be divided into two parts – a rectangle A with height and width and a triangle-like area B on top of the rectangle.

Chart

Description automatically generated

Graph A

Then, by Graph A, we have

as required.

1. By writing the natural logarithm function for some *x* as efine a sequence by Show that this sequence is decreasing and that for all

By looking at the difference between consecutive terms of the sequence, we have

Then, by properties of integration from Q3(a) we have

Thus, which means that is decreasing.

Since

and is decreasing,

The first term in the definition of the sequence, , can be seen as a left Riemann sum, which is an overestimation of the sequence since it is decreasing. Thus,

Thus, as required.

1. Why does exist?

Since is decreasing and bounded below, exists by the monotone convergence theorem, as required.